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(6) GENERALIZATION OF BILLINGSLEY'S INEQUALITIES[1]

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ABSTRACT

GENERALIZATION OF BILLINGSLEY'S INEQUALITIES

Let $\xi_1, \xi_2, \ldots, \xi_m$ be arbitrary random variables and define $S_k = \xi_1+\xi_2+\ldots+\xi_k$ for $1 \le k \le m$, $S_0 = 0$, $M_m = \max_{0\le k\le m} |S_k|$, $M'_m = \max_{0\le k\le m} \min(|S_k|, |S_m - S_k|)$ and $M''_m = \max_{0\le i\le j\le k\le m} \min(|S_j - S_i|, |S_k - S_j|)$. In this paper we establish bounds for the quantities $P\{M_m \ge \lambda\}$, $P\{M'_m \ge \lambda\}$ and $P\{M''_m \ge \lambda\}$ in terms of corresponding similar bounds assumed for the quantities $P\{|S_j - S_i| \ge \lambda\}$, $P\{|S_j - S_i| \ge \lambda,\ |S_k - S_j| \ge \lambda\}$, all $0 \le i \le j \le k \le m$. The bounds explicitly involve a nonnegative function $f(i,j)$ which is quasi-superadditive, i.e., $f(i,j) + f(j,k) \le Qf(i,k)$, all $0 \le i \le j \le k \le m$, for a fixed Q, $1 \le Q < 2$. The results generalize theorems of Billingsley (1968) for the case $Q = 1$ and $f(i,j) = \sum_{i<k\le j} u_k$, where $u_1, \ldots, u_m$ are nonnegative reals.

1. Introduction. Let $\xi_1, \xi_2, \ldots, \xi_m$ be arbitrary random variables. It is not assumed that the ξ_k's are independent or identically distributed. The only restrictions on the dependence will be those imposed by the assumed bounds on

$$P\{|S_j - S_i| \ge \lambda\},\ P\{|S_j - S_i| \ge \lambda,\ |S_k - S_j| \ge \lambda\},$$

where λ runs over an interval of the positive real line, and

$$S_j = \sum_{k=1}^{j} \xi_k \quad \text{for } 1 \le j \le m \text{ and } S_0 = 0.$$

Following Billingsley (1968, pp. 87-103) (we use the same notation that can be found there), define

$$M_m = \max_{0\le k\le m} |S_k|,$$

$$M'_m = \max_{0\le k\le m} \min(|S_k|, |S_m - S_k|),$$

$$M''_m = \max_{0\le i\le j\le k\le m} \min(|S_j - S_i|, |S_k - S_j|),$$

and

$$N_m = \min_{1\le \ell\le m} \max\{\max_{0\le i<\ell} |S_i|, \max_{\ell\le i\le m} |S_m - S_i|\}.$$

We recall that

$$M'_m \le M_m,\ M'_m \le M''_m, \text{ and } M''_m \le 2N_m.$$

Our main goal is to establish bounds for the quantities

$$P\{M_m \ge \lambda\},\ P\{M'_m \ge \lambda\},\ P\{M''_m \ge \lambda\}$$

in terms of corresponding similar bounds assumed for the quantities listed above. The bounds will be related in specific ways to the variables $S_j - S_i$, $0 \le i \le j \le m$, through some function $f(i,j)$ assumed to be nonnegative, nondecreasing in j for each fixed i, and Q-superadditive with $1 \le Q < 2$. The latter property was introduced by Móricz, Serfling and Stout (1981). A function $f(i,j)$, $0 \le i \le j \le m$, is said to be *quasi-superadditive with index* Q (or simply Q-*superadditive*) if

$$f(i,j) + f(j,k) \le Qf(i,k), \text{ all } 1 \le i \le j \le k \le m.$$

The case $Q = 1$ corresponds to the usual notation of superadditivity.

We note that there is a slight difference in notation between this paper and the paper mentioned above. The relation between the function $f(i,j)$ occurring here and the function $g(i,j)$ used there is the following: $f(i,j) = g(i+1,j)$ (and similarly, $S_j - S_i = S(i+1,j)$, the latter being also used there).

For later reference we collect the assumed properties of $f(i,j)$ as follows

(1.1a) $f(i,j) \geq 0$, $f(i,i) = 0$, all $0 \leq i \leq j \leq m$,

(1.1b) $f(i,j) \leq f(i,j+1)$, all $0 \leq i \leq j < m$,

(1.1c) $f(i,j) + f(j,k) \leq Qf(i,k)$, all $0 \leq i \leq j \leq k \leq m$.

In Billingsley's book (pp. 87-103) the case $f(i,j) = \sum_{i<k\leq j} u_k$ is treated, where $u_1, u_2, \ldots, u_m$ are nonnegative reals. This function $f(i,j)$ is clearly superadditive (even additive).

2. Main Results.

THEOREM 1. (The generalization of [1, Theorem 12.1].) *Let* $\alpha > 1/2$ *be a given real. Suppose that there exist a function* $f(i,j)$ *satisfying* (1.1) *with a* Q, $1 \leq Q < 2^{(2\alpha-1)/2\alpha}$, *and a* λ_0, $0 < \lambda_0 \leq +\infty$, *such that*

(2.1) $$P\{|S_j - S_i| \geq \lambda,\ |S_k - S_j| \geq \lambda\} \leq \frac{1}{\phi(\lambda)} f^{2\alpha}(i,k),\ \textit{all}\ 0 < \lambda < \lambda_0\ \textit{and}\ 0 \leq i \leq j \leq k \leq m,$$

where $\phi(\lambda) > 0$ *for* $0 < \lambda < \lambda_0$ *and for each constant* C, $0 < C < 1$, *we have*

(2.2) $$\inf_{0<\lambda<\lambda_0} \frac{\phi(C\lambda)}{\phi(\lambda)} = \chi(C) > 0; \quad \lim_{C\to 1-0} \chi(C) = 1.$$

Then there exists a constant $K \geq 1$, *depending on* α, Q *and* χ *but not on* m *or* $\{\xi_k\}$ *or otherwise on* f, *such that*

(2.3) $$P\{M'_m \geq \lambda\} \leq \frac{K}{\phi(\lambda)} f^{2\alpha}(0,m),\ \textit{all}\ 0 < \lambda < \lambda_0.$$

We note that $\phi(\lambda) = \lambda^{\gamma}$ satisfies condition (2.2) for each $\gamma \geq 0$. Actually, Theorem 1 was proved by Billingsley (1968) in the special case that $\phi(\lambda) = \lambda^{2\gamma}$, $\gamma \geq 0$, and $f(i,j) = \sum_{i<k\leq j} u_k$, $u_k \geq 0$. This remark pertains to the subsequent Theorem 2 and Corollaries 1 and 2.

The following corollary can be deduced from Theorem 1 in the same way that Theorem 12.2 is deduced from Theorem 12.1 in [1].

COROLLARY 1. (The generalization of [1, Theorem 12.2].) *Let* $\alpha > 1$ *be a given real. Suppose that there exist a function* $f(i,j)$ *satisfying* (1.1) *with a* Q, $1 \leq Q < 2^{(\alpha-1)/\alpha}$, *and a* λ_0, $0 < \lambda_0 \leq +\infty$, *such that*

$$P\{|S_j - S_i| \geq \lambda\} \leq \frac{1}{\phi(\lambda)} f^{\alpha}(i,j),\ \textit{all}\ 0 < \lambda < \lambda_0\ \textit{and}\ 0 \leq i \leq j \leq m,$$

where $\phi(\lambda) > 0$ *for* $0 < \lambda < \lambda_0$ *and* (2.2) *is satisfied for each* C, $0 < C < 1$. *Then there exists a constant* $K' \geq 1$, *depending on* α, Q *and* χ *but not on* m *or* $\{\xi_k\}$ *or otherwise on* f, *such that*

$$P\{M_m \geq \lambda\} \leq \frac{K'}{\phi(\lambda)} f^{\alpha}(0,m), \qquad 0 < \lambda < \lambda_0.$$

This result, using a direct proving method (and a somewhat different notation), was proved by Móricz, Serfling and Stout (1981, Theorem 3.1).

PROOF OF THEOREM 1. It goes along the same lines as the proof of [1, Theorem 12.1], i.e., by induction on m. The result is trivial for $m = 1$ and can be simply proved for $m = 2$.

Assume now as induction hypothesis that the result holds for each integer less than $m > 2$. We shall prove it for m itself. We may assume $f(0,m) > 0$. By (1.1), there exists an integer h, $1 \leq h \leq m$, such that

(2.4) $$f(0,h-1) \leq \frac{Q}{2} f(0,m) \leq f(0,h).$$

Then, by (1.1) and (2.4), we have

(2.5) $$f(h,m) \leq \frac{Q}{2} f(0,m).$$

Following Billingsley's proof, consider the next four quantities:

$$U_1 = \max_{0\le i\le h-1} \min\{|S_i|, |S_{h-1} - S_i|\},$$

$$U_2 = \max_{h\le j\le m} \min\{|S_j - S_h|, |S_m - S_j|\},$$

$$D_1 = \min\{|S_{h-1}|, |S_m - S_{h-1}|\},$$

$$D_2 = \min\{|S_h|, |S_m - S_h|\}.$$

By [1, (12.37) and (12.38)],

(2.6) $$P\{M'_m \ge \lambda\} \le P\{U_1 + D_1 \ge \lambda\} + P\{U_2 + D_2 \ge \lambda\}$$

and

(2.7) $$P\{U_1 + D_1 \ge \lambda\} \le P\{U_1 \ge p\lambda\} + P\{D_1 \ge q\lambda\},$$

where p and q are positive reals and $p+q = 1$.

By the induction hypothesis and (2.4), we have

$$P\{U_1 \ge p\lambda\} \le \frac{K}{\phi(p\lambda)} f^{2\alpha}(0,h-1) \le \frac{K}{\phi(p\lambda)} \frac{Q^{2\alpha}}{2^{2\alpha}} f^{2\alpha}(0,m).$$

By (2.1),

$$P\{D_1 \ge q\lambda\} \le \frac{1}{\phi(q\lambda)} f^{2\alpha}(0,m).$$

Taking (2.2) into account, from (2.7) it follows that

$$P\{U_1 + D_1 \ge \lambda\} \le \frac{Kf^{2\alpha}(0,m)}{\phi(t)}\left(\frac{1}{\chi(p)}\frac{Q^{2\alpha}}{2^{2\alpha}} + \frac{1}{K\chi(q)}\right).$$

The same inequality holds for U_2+D_2 (using (2.5) instead of (2.4)).

By (2.6), therefore,

$$P\{M'_m \ge \lambda\} \le \frac{Kf^{2\alpha}(0,m)}{\phi(t)}\left(\frac{1}{\chi(p)}\frac{Q^{2\alpha}}{2^{2\alpha-1}} + \frac{2}{K\chi(q)}\right).$$

By assumption $Q^{2\alpha}/2^{2\alpha-1} < 1$. Thus, thanks to (2.2), we can define p, $0 < p < 1$, in such a way that

$$\frac{1}{\chi(p)}\frac{Q^{2\alpha}}{2^{2\alpha-1}} < 1.$$

Then let $q = 1-p$ and define K by the condition

$$\frac{1}{\chi(p)}\frac{Q^{2\alpha}}{2^{2\alpha-1}} + \frac{2}{K\chi(q)} \le 1.$$

This completes the induction step and the proof of Theorem 1.

3. Further Inequalities.

THEOREM 2. (The generalization of [1, Theorem 12.5]) *Let* $\alpha > 1/2$ *be a given real. Suppose that there exist a function* $f(i,j)$ *satisfying* (1.1) *with a* Q, $1 \le Q < 2^{(2\alpha-1)/2\alpha}$, *and a* λ_0, $0 < \lambda_0 \le +\infty$, *such that condition* (2.1) *holds, where* $\phi(\lambda) > 0$ *for* $0 < \lambda < \lambda_0$ *and* (2.2) *is satisfied for each* C, $0 < C < 1$. *Then there exists a constant* $K'' \ge 1$, *depending on* α, Q *and* χ *but not on* m *or* $\{\xi_k\}$ *or otherwise on* f, *such that we have both*

$$P\{M''_m \ge \lambda\} \le \frac{K''}{\phi(\lambda)} f^{2\alpha}(0,m), \text{ all } 0 < \lambda < \lambda_0,$$

and

$$P\{M_m \ge \lambda\} \le \frac{K''}{\phi(\lambda)} f^{2\alpha}(0,m), \text{ all } 0 < \lambda < \lambda_0.$$

The proof of Theorem 2 closely follows that of [1, Theorem 12.5], using the same modifications we performed in the proof of Theorem 1. We do not enter into details.

COROLLARY 2. (The generalization of [1, Theorem 12.6]) *Let* $\alpha > 1/2$ *be a given real. Suppose that there exist a function* $f(i,j)$ *satisfying* (1.1) *with* $Q = 1$ *and a* λ_0, $0 < \lambda_0 \le +\infty$, *such that*

(3.1) $$P\{|S_j - S_i| \ge \lambda, |S_k - S_j| \ge \lambda\} \le \frac{1}{\phi(\lambda)} f^{\alpha}(i,j)\, f^{\alpha}(j,k),$$

all $0 < \lambda < \lambda_0$ *and* $0 \le i \le j \le k \le m$, *where* $\phi(\lambda) > 0$ *for* $0 < \lambda < \lambda_0$ *and* (2.2) *is satisfied for each* C, $0 < C < 1$. *Then there exists a constant*

$K''' \geq 1$, ***depending on*** σ ***and*** χ ***but not on*** m ***or*** $\{\xi_k\}$ ***or otherwise on*** f, ***such that we have both***

$$P\{M''_m \geq \lambda\} \leq \frac{K'''}{\phi(\lambda)} f^{2\alpha}(0,m) \min_{1\leq h\leq m} \left[1 - \frac{f(h-1,h)}{f(0,m)}\right]^{\alpha} \tag{3.2}$$

and

$$P\{N_m \geq \lambda\} \leq \frac{K'''}{\phi(\lambda)} f^{2\alpha}(0,m) \min_{1\leq h\leq m} \left[1 - \frac{f(h-1,h)}{f(0,m)}\right]^{\alpha} \tag{3.3}$$

for all $0 < \lambda < \lambda_0$.

PROOF OF COROLLARY 2. Choose h to minimize the final factor in (3.3). Following [1, (12.70), (12.71) and (12.74)] write

$$A_1 = \min_{1\leq \ell_1 < h} \max\{\max_{0\leq i<\ell_1} |S_i|, \max_{\ell_1\leq i<h} |S_{h-1} - S_i|\},$$

$$A_2 = \min_{h<\ell_2\leq m} \max\{\max_{h\leq i\leq \ell_2} |S_i - S_h|, \max_{\ell_2\leq i\leq m} |S_m - S_i|\},$$

and

$$B = \max\{\mu(0,h-1,m);\ \mu(0,h-1,h);\ \mu(h-1,h,m);\ \mu(0,h,m)\},$$

where

$$\mu(i,j,k) = \min\{|S_j - S_i|,\ |S_k - S_j|\}.$$

Since (3.1) implies (2.1), by virtue of Theorem 2 we have

$$P\{A_1 \geq \lambda\} \leq \frac{K''}{\phi(\lambda)} f^{2\alpha}(0,h-1)$$

and

$$P\{A_2 \leq \lambda\} \leq \frac{K''}{\phi(\lambda)} f^{2\alpha}(h,m).$$

Owing to (1.1) for $Q = 1$,

$$f(0,h-1) \leq f(0,h) - f(h-1,h) \leq f(0,m) - f(h-1,h)$$

and

$$f(h,m) \leq f(0,m) - f(0,h) \leq f(0,m) - f(h-1,h).$$

Combining the last two inequalities with the two preceding ones, we obtain

$$P\{A_1 \geq \lambda\} \leq \frac{K''}{\phi(\lambda)} f^{2\alpha}(0,m) \left[1 - \frac{f(h-1,h)}{f(0,m)}\right]^{\alpha} \tag{3.4}$$

and

$$P\{A_2 \geq \lambda\} \leq \frac{K''}{\phi(\lambda)} f^{2\alpha}(0,m) \left[1 - \frac{f(h-1,h)}{f(0,m)}\right]^{\alpha} \tag{3.5}$$

Finally, due to (3.1),

$$P\{\mu(0,h-1,m) \geq \lambda\} \leq \frac{1}{\phi(\lambda)} f^{\alpha}(0,h-1)\, f^{\alpha}(h-1,m) \leq \frac{f^{2\alpha}(0,m)}{\phi(\lambda)} \left[1 - \frac{f(h-1,h)}{f(0,m)}\right]^{\alpha},$$

and there is a similar inequality for each of the other μ's occurring in B. Therefore,

$$P\{B \geq \lambda\} \leq \frac{4f^{2\alpha}(0,m)}{\phi(\lambda)} \left[1 - \frac{f(h-1,h)}{f(0,m)}\right]^{\alpha}. \tag{3.6}$$

By [1, (12.76)],

$$N_m \leq \max\{A_1, A_2\} + 2B;$$

consequently,

$$P\{N_m \geq \lambda\} \leq P\{A_1 \geq \tfrac{1}{2}\lambda\} + P\{A_2 \geq \tfrac{1}{2}\lambda\} + P\{B \geq \tfrac{1}{4}\lambda\}.$$

Applying inequalities (3.4), (3.5) and (3.6), and taking (2.2) into account, we have

$$P\{N_m \geq \lambda\} \leq \left(\frac{2K''}{\chi(\frac{1}{4})} + \frac{4}{\chi(\frac{1}{4})}\right) \frac{f^{2\alpha}(0,m)}{\phi(\lambda)} \left[1 - \frac{f(h-1,h)}{f(0,m)}\right]^{\alpha}.$$

This is the desired inequality (3.3).

Hence (3.2) immediately follows since $N_m \leq 2M''_m$.

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20. ABSTRACT

Let $\xi_1, \xi_2, \ldots, \xi_m$ be arbitrary random variables and define $S_k = \xi_1 + \xi_2 + \ldots \xi_k$ for $1 \le k \le m$, $S_0 = 0$, $M_m = \max_{0 \le k \le m} |S_k|$, $M'_m = \max_{0 \le k \le m} \min(|S_k|, |S_m - S_k|)$ and $M''_m = \max_{0 \le i \le j \le k \le m} \min(|S_j - S_i|, |S_k - S_j|)$. In this paper we establish bounds for the quantities $P(M_m \ge \lambda)$, $P(M'_m \ge \lambda)$ and $P(M''_m \ge \lambda)$ in terms of corresponding similar bounds assumed for the quantities $P(|S_j - S_i| \ge \lambda)$, $P(|S_j - S_i| \ge \lambda, |S_k - S_j| \ge \lambda)$, all $0 \le i \le j \le k \le m$. The bounds explicitly involve a nonnegative function $f(i,j)$ which is quasi-superadditive, i.e., $f(i,j) + f(j,k) \le Qf(i,k)$, all $0 \le i \le j \le k \le m$, for a fixed Q, $1 \le Q < 2$. The results generalize theorems of Billingsley (1968) for the case $Q = 1$ and $f(i,j) = \sum_{i<k\le j} u_k$, where $u_1, \ldots, u_m$ are nonnegative reals.